The Classical Laplace Plane and its use as a Stable Disposal Orbit for GEO

Aaron J. Rosengren, Daniel J. Scheeres, and Jay W. McMahon
University of Colorado, Boulder, United States

ABSTRACT

The classical Laplace plane is a frozen orbit, or equilibrium solution for the averaged dynamics arising from Earth oblateness and lunisolar gravitational perturbations. The pole of the orbital plane of uncontrolled GEO satellites regress around the pole of the Laplace plane at nearly constant inclination and rate. In accordance with Friesen et al. (1993), we show how the this stable plane can be used as a robust long-term disposal orbit. The current graveyard regions for end-of-life retirement of GEO payloads, which is several hundred kilometers above GEO, cannot contain the newly discovered high area-to-mass ratio debris population. Such objects are highly susceptible to the effects of solar radiation pressure exhibiting dramatic variations in eccentricity and inclination over short periods of time. The Laplace plane graveyard, on the contrary, would trap this debris, analogous to the ocean gyres, and would not allow these objects to rain down through GEO. Since placing a satellite in this inclined orbit is quite expensive, we discuss some alternative disposal schemes that have acceptable cost-to-benefit ratios.

1. INTRODUCTION

Nearly a half century has elapsed since satellites were first launched into the geostationary (equatorial, circular-synchronous) orbit — the altitude of 35,786 km where satellites appear to remain fixed over a single point on the equator throughout the day, providing a unique vantage point for communication, meteorology, science, and military applications. The geostationary ring is the most susceptible region to space debris because there is no natural cleansing mechanism to limit the lifetimes of the debris at this altitude. Only objects in low-altitude orbits will return to Earth without human intervention through the influence of atmospheric drag, which steadily reduces their orbital energy until they burn up within the atmosphere. In some preferential low Earth orbit (LEO) regions, where the population is above a critical spatial density, random collisions are predicted to produce new debris at a rate that is greater than the removal rate due to orbital decay (the Kessler syndrome) [12, 15]. In GEO, the relative velocities are much lower (less than 1 km/s), meaning that the the damage done by impact and the amount of detritus generated in a collision is not as severe. However, because debris would contaminate this unique and valuable resource practically forever, placing satellites in super-synchronous disposal orbits at the ends of their operational lifetimes has been recommended and practiced as one possible means of protecting this orbital environment [8, 9]. These satellites are also passivated to reduce the probability of future explosions by removing any on-board stored energy, such as residual fuel or pressurants and charged batteries.

The high area-to-mass ratio (HAMR) debris population in GEO space, discovered through optical observations by Schildknecht and colleagues in 2004, demonstrates that energetic breakups and collisions are not the only source of concern [24, 25]. This hitherto unknown class of body — having area-to-mass ratios hundreds or thousands of times greater than that of a typical satellite and is thus strongly perturbed by solar radiation pressure (SRP) — has been linked to aging satellites in the storage orbits. Such objects can be generated in a variety of ways: material deterioration, surface degradation, collisions, and explosions, to name just a few. The low energy release of HAMR objects from aging satellites abandoned in disposal orbits is not directly addressed in the international policies that established the graveyard [8, 11]. The current disposal region, which is several hundred kilometers above GEO, is not well suited as a graveyard for two important reasons: it does not mitigate the possibility of collisions between the uncontrolled objects residing in this region, and it cannot contain the HAMR population.

In geostationary orbit, stationkeeping maneuvers, regular rocket firings, are required to constantly maintain the orderly arrangements of operational satellites. The orbital dynamics of uncontrolled geostationary satellites is governed by the oblateness (equatorial bulge) of the Earth and third-body gravitational interactions induced by the Sun and the Moon (lunisolar perturbations). By itself, Earth's oblateness causes the pole of the orbital plane to precess around

Earth's rotation pole. Lunisolar perturbations, if acting alone, will have a similar effect, but the precession will now take place about the poles of the orbital planes of the Moon and the Sun, respectively. The motion of the orbit pole of the satellite is a combination of simultaneous precession about these three different axes, one of which, the pole of the Moon's orbit, regresses around the pole of the ecliptic with a period of 18.61 years. The classical Laplace plane is the mean reference plane about whose axis the satellite's orbit precesses [1, 26].

A circular orbit in the classical Laplace plane experiences no secular precession; thus, the Laplace plane is simply a frozen orbit, or equilibrium solution for the averaged dynamics arising from the quadrupole potential of the gravitational perturbations [26]. Under the assumption that the lunar orbit lies in the ecliptic, Allan and Cook found an approximate Laplace plane at GEO, which lies between the plane of the Earth's equator and that of the ecliptic and passes through their intersection (the vernal equinox), and which has an inclination of about $7\overset{\circ}{.}2$ with respect to the equator [1]. The geostationary satellites, following cessation of active stationkeeping, precess at a constant inclination about the pole of the Laplace plane with a period of nearly 53 years [3, 4, 25]. Sufficient time has now passed for the orbits of the earliest uncontrolled satellites to complete their long-period motion, as indicated by Schildknecht [25].

In the early 1990s, when the problems of overcrowding of geostationary orbit with operational and defunct satellites began to emerge, Friesen et al. suggested the use of the classical Laplace plane for satellite applications and as an orbital debris management strategy for GEO orbit [3, 4]. The orbit plane of a geosynchronous satellite with such an orientation, notes Otis Graf, would be fixed in space [6]. The significance of the Laplace plane for use as a GEO disposal orbit is that the orbits of satellites placed in this stable equilibrium will be fixed on average, and that any orbit at small inclination to it regresses around this plane at nearly constant inclination and rate [1]. This stable graveyard can be specified for a range of semi-major axes above (or below) GEO, and satellites placed in this region will have drastically reduced relative encounter velocities, compared to the conventional graveyard orbits [4]. Thus, if collisions were to occur between satellites in the stable graveyard, they would occur at very low velocities, thereby damping out the relative motion of these objects and keeping them in this stable plane.

In a recent paper, we investigated the robustness of the Laplace plane graveyard orbit to the recently discovered HAMR debris [22]. We rigorously showed how solar radiation pressure modifies the classical Laplace plane. Based on this analysis, we found that if satellites located in the classical Laplace plane graveyard orbit shed HAMR objects, they will be trapped in inclination and ascending node phase space, and will not likely interfere with the geostationary satellites. This current paper reviews these results, examines the robustness of these solutions to more realistic SRP models, and discusses the economic viability of this proposed GEO graveyard.

2. FROZEN ORBITS IN THE EARTH-MOON-SUN SYSTEM

2.1. The Classical Laplace Plane

The orbital geometry can be described naturally and succinctly in terms of the angular momentum and eccentricity vectors (i.e., the Milankovitch elements [21]):

$$\boldsymbol{H} = \widetilde{\boldsymbol{r}} \cdot \boldsymbol{v}, \tag{1}$$

$$\boldsymbol{e} = \frac{1}{\mu}\widetilde{\boldsymbol{v}} \cdot \boldsymbol{H} - \frac{\boldsymbol{r}}{|\boldsymbol{r}|}, \tag{2}$$

where $\boldsymbol{r}$ and $\boldsymbol{v}$ are the position and velocity vectors, and the notation $\widetilde{\boldsymbol{a}}$ denotes the cross-product dyadic, defined such that $\widetilde{\boldsymbol{a}} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \widetilde{\boldsymbol{b}} = \boldsymbol{a} \times \boldsymbol{b}$ for any vectors $\boldsymbol{a}, \boldsymbol{b}$ in $\boldsymbol{R}^3$. These vectors have a clear geometrical significance: $\boldsymbol{H}$ points perpendicular to the instantaneous orbit plane and has magnitude $H = \sqrt{\mu a(1-e^2)}$; $\boldsymbol{e}$ defines the orientation of the major axis in the orbital plane, pointing towards the instantaneous periapsis of the orbit, and its magnitude is the eccentricity. Recall the basic definition of these vectors in terms of the Keplerian orbital elements:

$$\boldsymbol{H} = H\hat{\boldsymbol{h}} = H(\sin i \sin\Omega \hat{\boldsymbol{x}} - \sin i \cos\Omega \hat{\boldsymbol{y}} + \cos i \hat{\boldsymbol{z}}), \tag{3}$$

$$\begin{aligned} \boldsymbol{e} = e\hat{\boldsymbol{e}} \;\; &= e[(\cos\omega\cos\Omega - \cos i \sin\omega \sin\Omega)\hat{\boldsymbol{x}} \\ &\quad + (\cos\omega\sin\Omega + \cos i \sin\omega\cos\Omega)\hat{\boldsymbol{y}} \\ &\quad + \sin i \sin\omega \hat{\boldsymbol{z}}], \end{aligned} \tag{4}$$

where μ is the gravitational parameter, a the semi-major axis, e the eccentricity, i the inclination, Ω the right ascension of the ascending node, and ω the argument of periapsis. As the relationship between the classical orbit elements and $\boldsymbol{H}$ and $\boldsymbol{e}$ are complex, we find it simpler to rely on the geometric definition of these vectors.

In secular dynamics, the semi-major axis is fixed and the problem can be reduced to understanding the evolution of the scaled angular momentum vector $\boldsymbol{h} = \boldsymbol{H}/\sqrt{\mu a}$ and the eccentricity vector $\boldsymbol{e}$. Considering only gravitational perturbations and assuming that the lunar orbit lies in the ecliptic, the secular equations of motion arising from Earth oblateness and lunisolar perturbations can be stated as [1, 22, 26]

$$\dot{\boldsymbol{h}} = -\frac{\omega_2}{h^5}(\hat{\boldsymbol{p}}\cdot\boldsymbol{h})\widetilde{\hat{\boldsymbol{p}}}\cdot\boldsymbol{h} - (\omega_m+\omega_s)\hat{\boldsymbol{H}}_s\cdot(5\boldsymbol{e}\boldsymbol{e}-\boldsymbol{h}\boldsymbol{h})\cdot\widetilde{\hat{\boldsymbol{H}}}_s, \tag{5a}$$

$$\dot{\boldsymbol{e}} = -\frac{\omega_2}{2h^5}\left\{\left[1-\frac{5}{h^2}(\hat{\boldsymbol{p}}\cdot\boldsymbol{h})^2\right]\widetilde{\boldsymbol{h}} + 2(\hat{\boldsymbol{p}}\cdot\boldsymbol{h})\widetilde{\hat{\boldsymbol{p}}}\right\}\cdot\boldsymbol{e} - (\omega_m+\omega_s)\left[\hat{\boldsymbol{H}}_s\cdot(5\boldsymbol{e}\boldsymbol{h}-\boldsymbol{h}\boldsymbol{e})\cdot\widetilde{\hat{\boldsymbol{H}}}_s - 2\widetilde{\boldsymbol{h}}\cdot\boldsymbol{e}\right],$$

in which $\hat{\boldsymbol{p}}$ is a unit vector aligned with the maximum axis of inertia of the Earth (i.e., Earth's rotation pole), $\hat{\boldsymbol{H}}_s$ is a unit vector aligned with the angular momentum vector of the Sun's orbit (i.e., Earth's orbit pole), and

$$\omega_2 = \frac{3nJ_2R^2}{2a^2}, \quad \omega_p = \frac{3\mu_p}{4na_p^3(1-e_p^2)^{3/2}}, \tag{6}$$

where $n = \sqrt{\mu/a^3}$ is the satellite's mean motion, J_2 is the oblateness gravity field coefficient, R is the mean equatorial radius of the Earth, and μ_p, a_p, and e_p are the gravitational parameter, semi-major axis, and eccentricity, respectively, of the perturbing body (Moon or Sun).

Tremaine et al. (2009) defines the Laplace equilibria to be stationary solutions solutons of Eq. 5, or orbits where the average angular momentum and eccentricity vectors remain constant [26]. There are five types of equilibria for the system, each classified by the orientation of the vectors $\boldsymbol{h}$ and $\boldsymbol{e}$. For an initially circular orbit, $\dot{\boldsymbol{e}}$ is identically zero and the orbit will remain circular throughout. There are two kinds of equilibria in this case: the polar Laplace equilibrium and the classical Laplace plane. See [26] for details on the stability of the polar Laplace equilibrium; its application to Earth orbiters is given in [13] and [27]. We consider only the classical Laplace plane equilibrium which is defined as the circular Laplace equilibria ($\boldsymbol{e} = \boldsymbol{0}$) for which $\boldsymbol{h}$ lies in the principal plane specified by the vectors $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{H}}_s$. It may be specified by an azimuthal angle φ, as shown in Fig. 1, where ε is the obliquity of the ecliptic (*i.e.*, the angle between the Earth's equatorial plane and orbit). The frozen orbit condition becomes [1, 22, 26]

$$\omega_2\sin 2\varphi + (\omega_m+\omega_s)\sin 2(\varphi-\varepsilon) = 0 \quad \text{or}$$
$$\tan 2\varphi = \frac{\sin 2\varepsilon}{\cos 2\varepsilon + (r_L/a)^5}, \tag{7}$$

where r_L is the Laplace radius defined by

$$r_L^5 = a^5\frac{\omega_2}{\omega_m+\omega_s}. \tag{8}$$

The Laplace radius is the critical distance at which the Laplace plane lies halfway between the equatorial and ecliptic planes: it is thus the geocentric distance where the effects of oblateness and lunisolar forces are equal. Note that for Earth-orbital dynamics, this equilibrium is stable to both changes in the orbit plane orientation and eccentricity [26].

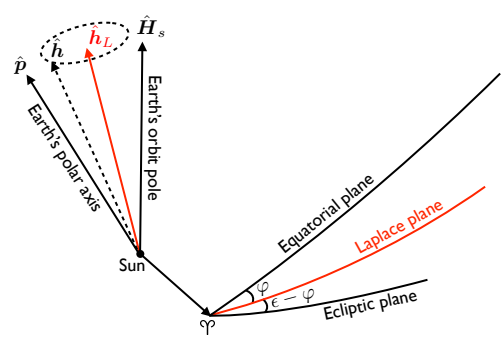
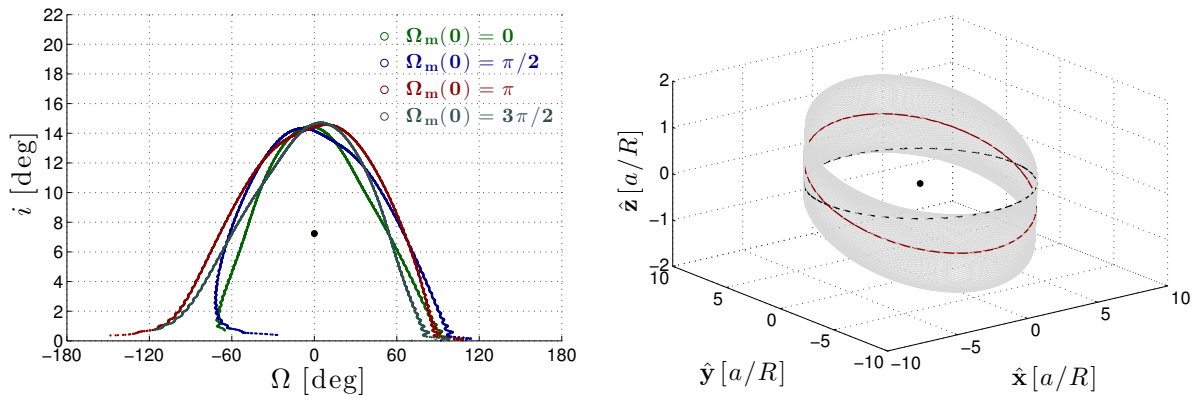


Fig. 1. Geometry of the Laplace plane equilibrium. The pole $\hat{\boldsymbol{h}}$ of any orbit at small inclination to the pole of the Laplace plane will regress about $\hat{\boldsymbol{h}}_L$ at constant inclination and rate.

Shown in Fig. 2 is the long-term evolution of the inclination and ascending node, in the Earth equatorial frame, of initially geostationary satellites. These (uncontrolled) satellites precess at a nearly constant inclination about the pole of the Laplace plane with a period of about 53 years. The orbital planes of these objects evolve in a predictable way; that is, their inclinations and ascending nodes are strongly correlated (see Fig. 2(a)). In fact, the former Soviet Union designed their geostationary satellite constellation to take advantage of this systematic structure: by selecting the initial inclination and ascending node such that the perturbations will naturally reduce the inclination to zero before increasing again, the satellite's inclination is kept below a few degrees over its lifetime without the need of expensive north-south stationkeeping [10, 11].



(a) Scatter plot of the time-series, over 53 years, of inclination and ascending node as predicted by an averaged model [20]. The classical Laplace plane has $i = 7.^\circ 2$ and $\Omega = 0^\circ$ as indicated by the black dot.

(b) Three-dimensional picture of the evolution of one of the satellites. The Earth is at the center, the initial geostationary orbit is indicated by the black orbit, and the classical Laplace plane is shown in red. The Laplace plane is the plane of symmetry for the evolution.

Fig. 2. Long-term motion, in the Earth-equatorial frame, of the orbital plane of initially geostationary satellites. A numerical integration was performed over 53 years of an averaged model developed in [20] for four different initial positions of the lunar orbit, i.e., four different launch dates as follows: July 1964 ($\Omega_m(0) = \pi/2$), March 1969 ($\Omega_m(0) = 0$), November 1973 ($\Omega_m(0) = 3\pi/2$), July 1978 ($\Omega_m(0) = \pi$).

2.2. The Modified Laplace Plane

Solar radiation pressure is the largest non-gravitational perturbative force to affect the motion of HAMR objects in high-Earth orbits, causing extreme changes in their orbital parameters over short time periods. From a previously derived solution for the secular motion of an orbiter about a small body (asteroid or comet) in a solar radiation pressure dominated environment [19, 23], we have found that SRP acting alone will have a precisely similar secular effect on initially circular orbits as solar gravitational perturbations, causing the orbit to precess around the pole of the ecliptic in accordance with the prediction of Allan and Cook [2], but with the rate of rotation being $(1-\cos\Lambda)/\cos\Lambda$, where Λ is the SRP perturbation angle defined as [17].

$$\tan\Lambda = \frac{3\beta}{2V_{lc}H_s}, \tag{9}$$

in which $\beta = (1+\rho)(A/m)P_\Phi$, ρ is the total reflectance or albedo of the body, A/m is the appropriate cross-sectional area-to-mass ratio in m^2/kg, P_Φ is the solar radiation constant ($\sim 1\times 10^8$ kg km^3/s^2/m^2), V_{lc} is the local circular speed of the object about the Earth, and H_s is the specific angular momentum of the Earth about the Sun. Note that as the perturbation becomes strong, $\Lambda \to \pi/2$; and as it becomes weak $\Lambda \to 0$.

Including the secular effect of SRP into Eq. (5), we can find an approximate mean pole around which the orbit precesses. The condition for the modified equilibrium becomes

$$\omega_2(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{h}})\widetilde{\hat{\boldsymbol{p}}}\cdot\hat{\boldsymbol{h}} + (\omega_m+\omega_s)(\hat{\boldsymbol{H}}_s\cdot\hat{\boldsymbol{h}})\widetilde{\hat{\boldsymbol{H}}}_s\cdot\hat{\boldsymbol{h}} + \omega_{srp}\widetilde{\hat{\boldsymbol{H}}}_s\cdot\hat{\boldsymbol{h}} = \mathbf{o} \quad \text{or} \tag{10}$$

$$\omega_2\sin 2\varphi + (\omega_m+\omega_s)\sin 2(\varphi-\varepsilon) + 2\omega_{srp}\sin(\varphi-\varepsilon) = 0, \tag{11}$$

where

$$\omega_{srp} = \frac{2\pi(1-\cos\Lambda)}{T_s \cos\Lambda}, \tag{12}$$

and T_s is the Earth's orbital period in seconds.

The orientation of the modified Laplace surface between the degenerate states (equatorial and ecliptic) is given as a function of a/R in Fig. 3. Solar radiation pressure modifies the classical Laplace plane, increasing its inclination relative to the equator with increasing Λ; each HAMR objects has its own modified Laplace plane for a given semi-major axis and effective area-to-mass ratio (or corresponding modified Laplace surface).

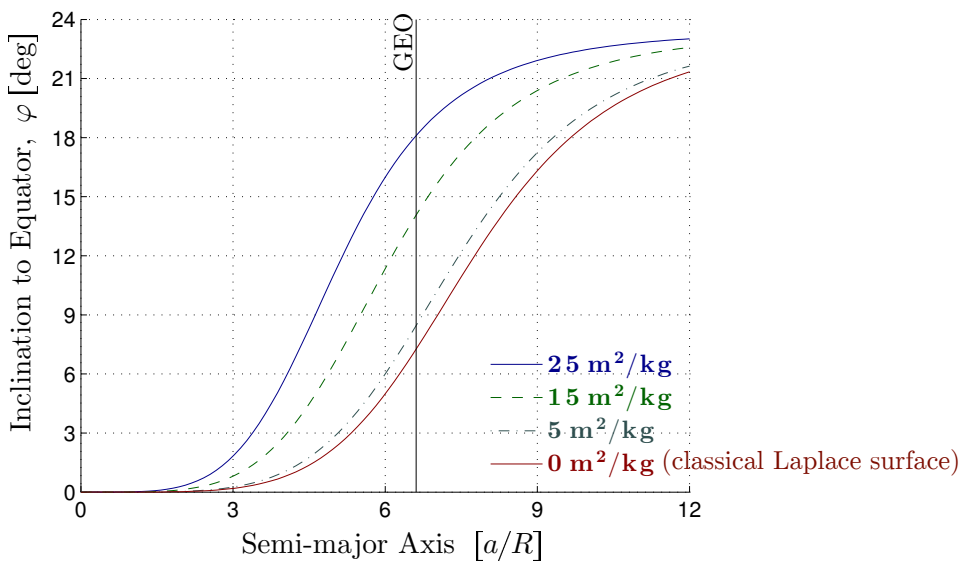


Fig. 3. Inclination of the Laplace plane equilibrium relative to the Earth's equator as a function of semi-major axis in Earth radii for a range of HAMR values. For a object near the Earth, the Laplace plane (both classical and modified) lies approximately in the Earth's equatorial plane, while for distant objects, it coincides with the ecliptic plane; all three planes sharing a common node (the vernal equinox). Between these two degenerate states, the Laplace plane at a given semi-major axis lies at some intermediate orientation, generating the warped Laplace surface. Note that an object with $A/m = 0$ m^2/kg corresponds to the classical Laplace surface.

3. THE LAPLACE PLANE GRAVEYARD ORBIT

The current disposal orbit scheme, established by the Inter-Agency Space Debris Coordination Committee and supported by the International Telecommunication Union, is to boost retired satellites into super-synchronous orbits several months before station-keeping fuel is expected to be exhausted [8, 9]. The minimum altitude threshold for re-orbiting incorporates the geostationary protected region (i.e., the operational station-keeping zone and maneuver corridor) as well as an allowance for gravitational and non-gravitational perturbations, as shown in Fig. 4. The stability of the super-synchronous disposal orbits and their potential to reduce collision hazards have been investigated extensively in the literature [7, 11]; however, these studies have focused on long-term simulations of intact satellites, which have very low area-to-mass ratios. We have found that this current disposal scheme for end-of-life retirement of GEO payloads is not well suited as a long-term graveyard because it does not mitigate the possibility of collisions between the uncontrolled objects residing in this region, nor does it reduce the severity of such collisions, and it cannot contain the high area-to-mass ratio debris population [22, 25]. The long-term evolution of the orbital plane of HAMR objects, released from super-synchronous disposal orbits, is shown in Fig. 5(a). An increase in Λ results in a faster and wider clockwise precession of the orbit pole [3], with the secular precession period as a function of Λ given in Figure 6. When HAMR objects return to zero inclination, which occurs much faster than the uncontrolled satellites, they may very rapidly cross the geostationary protected region due to their significant eccentricity growth (q.v., [20]).

For these reasons, we reconsider the possibility of using the stable Laplace plane equilibrium as a robust GEO graveyard, a notion originally put forward by Friesen et al. (1993), but which has not been fully appreciated in the scholarly world. Not only will satellites orbiting in this region have drastically reduced relative encounter velocities, thereby reducing the likelihood of a collisional cascade [4], but we have found that this region is robust to large SRP perturbations [22]. In particular, if satellites located in the classical Laplace plane graveyard orbit shed HAMR objects, they will be trapped in inclination and ascending node phase space, and will not likely cross the GEO protected region, as demonstrated in Figure 5(b).

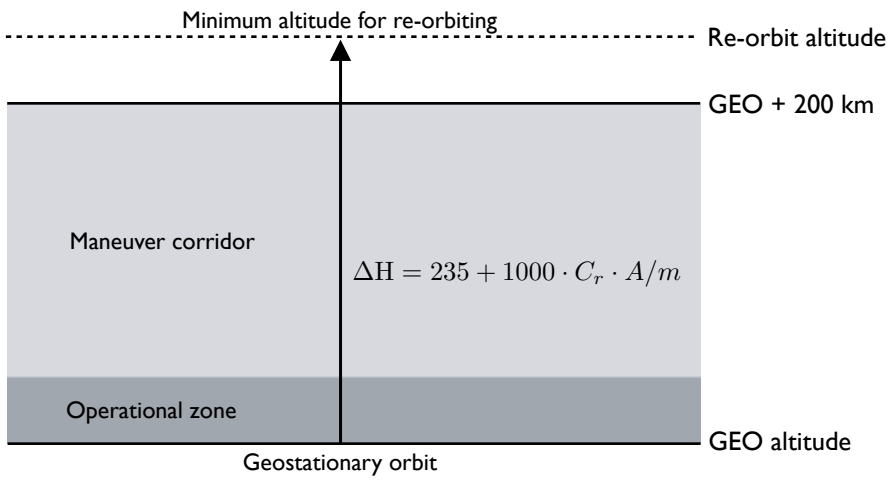
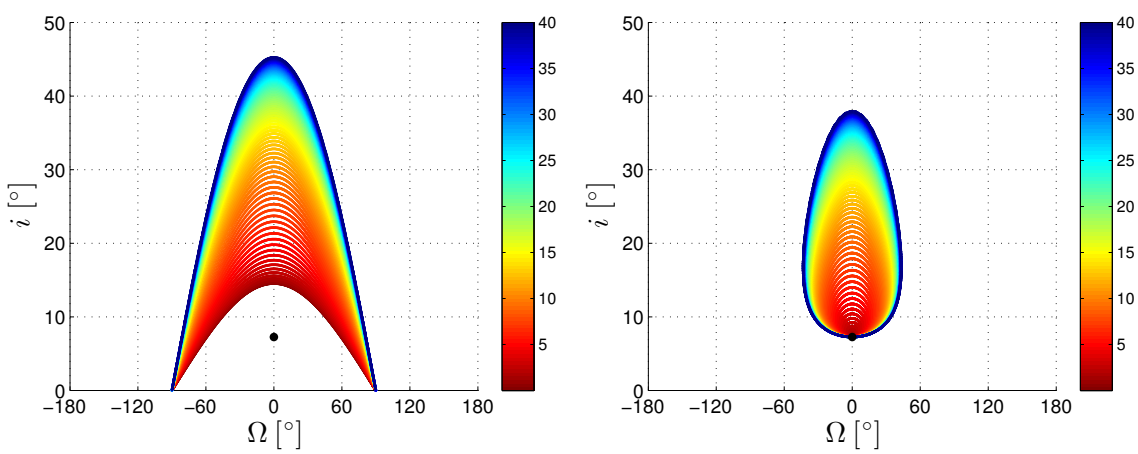


Fig. 4. The current, internationally established, disposal scheme for end-of-life retirement of GEO payloads. The accepted re-orbiting altitude is specified by ΔH, which accounts for the geostationary protected region and an allowance for perigee oscillation due to gravitational and non-gravitational perturbations (Adapted from [9].)



(a) Super-synchronous disposal orbit.

(b) Laplace plane disposal orbit.

Fig. 5. Qualitative evolution of the orbital planes of HAMR objects, released from geostationary orbit and from the classical Laplace plane illustrating the nature of the problem. The colorbar indicates the value of Λ and the position of the classical Laplace plane is indicated by the black dot.

3.1. Robustness of Laplace Graveyard to High-Fidelity SRP Models

The solution to the modified Laplace plane equilibrium and its implication to the high area-to-mass ratio debris are based on what is known in astrodynamical parlance as the cannonball model of solar radiation pressure, which treats

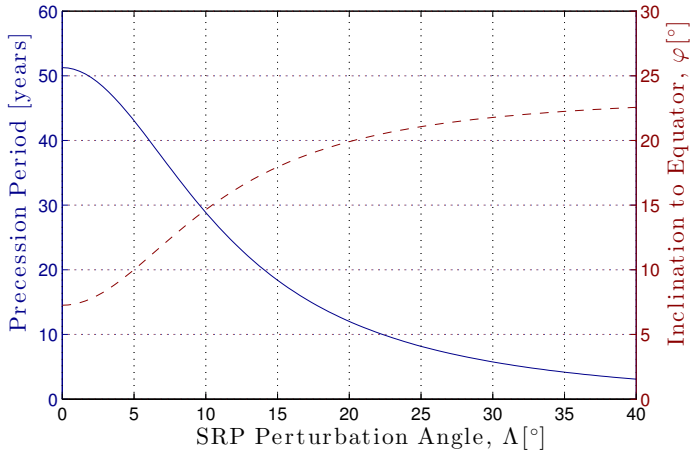


Fig. 6. Approximate secular precession period of the orbital pole as a function of Λ, and inclination of the modified Laplace plane equilibrium. The pole of the orbit precesses at constant rate and inclination around the approximate mean pole $\hat{\boldsymbol{h}}_L$ with a period $2\pi/\omega$, where $\omega\hat{\boldsymbol{h}}_L = \omega_2\hat{\boldsymbol{p}} + (\omega_m + \omega_s + \omega_{srp})\hat{\boldsymbol{H}}_s$. The modified Laplace plane is the plane of symmetry for the HAMR objects of Fig. 5.

the object as a sphere with constant optical properties. The net acceleration is assumed to be acting along the object-Sun line, which is taken to be parallel to the Earth-Sun line, and the total momentum transfer from the incident solar photons is modeled as insolation plus reflection. This attitude-independent model neglects any force component normal to the Earth-Sun line that results from an aspherical shape or nonuniformly reflecting surface. In some cases, depending on the rotational motion of the object, this non-radial component of the radiation pressure is negligible or will average out over time periods that are small compared to the orbital period. However, the validity of this frequently made assumption for debris, and HAMR objects in particular, may not be well established [5, 16].

The direction of SRP acceleration is not, in general, directly away from the Sun, as indicated by the cannonball model. Rather, this direction depends on the shape of the object, its optical properties, and its orientation with respect to the Earth-Sun line. The total radiation pressure of the incident sunlight, assuming that the Sun acts as a point source, can be modeled to first order as

$$P(d_s) = \frac{P_\Phi}{d_s^2}, \tag{13}$$

where d_s is the distance between the Earth and the Sun. The net acceleration due to the solar photons can be written in the general form [21]

$$\boldsymbol{a}_{srp} = \frac{\beta'}{d_s^2}\hat{\boldsymbol{a}}, \tag{14}$$

in which $\hat{\boldsymbol{a}}$ is the net direction of the acceleration and

$$\beta' = \frac{P_\Phi \left|\sum_{i=1}^{N} \boldsymbol{F}_i\right|}{m}, \tag{15}$$

where $\boldsymbol{F}_i$ represents the solar radiation force acting on a unit area A and m is the total mass of the body. Accounting for the total momentum transfer of the solar photons striking and recoiling off the surface element of a general body, $\boldsymbol{F}_i$ can be specified as

$$\boldsymbol{F}_i = -P(d_s)\left[\{\rho s(2\hat{\boldsymbol{n}}\hat{\boldsymbol{n}} - \boldsymbol{U}) + \boldsymbol{U}\} \cdot \hat{\boldsymbol{u}}\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}} + \{B(1-s)\rho + (1-\rho)B\}\,\hat{\boldsymbol{n}}\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{u}}\right] H(\hat{\boldsymbol{u}})A \tag{16}$$

where s is the fraction of specularly reflected light, $\hat{\boldsymbol{n}}$ is the unit normal to the surface, $\hat{\boldsymbol{u}} = \hat{\boldsymbol{d}}_s - \boldsymbol{r}$ is the unit vector from the surface to the Sun, $\hat{\boldsymbol{d}}_s$ is the position vector of the Sun relative to the Earth, B is a scattering coefficient that describes the fraction of light scattered normal to the surface (equal to 2/3 for an ideal Lambertian surface), and $H(\boldsymbol{u})$ is the visibility function for the surface and is equal to 1 when the Sun is in view and 0 otherwise.

As there is no method to incorporate this physically realistic SRP model with a lack of a priori information (i.e., object geometry, attitude behavior, surface properties, thermal characteristics, etc.), and because a systematic study of the entire parameter space represents an insurmountable task, several simplifications must be made. We have shown in [21] that if the net direction in which the SRP acceleration acts lies within the Earth's heliocentric orbit plane, the object will have similar dynamics to a cannonball. Therefore, the main non-cannonball effects are associated with the out-of-plane component of solar radiation pressure. For simplicity, consider a flat plate object which maintains a fixed orientation with respect to the Earth-Sun line. The disturbing acceleration — in the case of perfect specular reflection ($\rho s = 1$) — can be represented as

$$\boldsymbol{a}_{srp} = -\frac{2(A/m)P_{\Phi}\left(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{d}}_s\right)^2}{d_s^2}\hat{\boldsymbol{n}} \tag{17}$$

where we assume that the object is close to the Earth, or $r \ll d_s$, and we ignore the possible effect of the object passing through the Earth's shadow . The surface normal direction can be specified as

$$\hat{\mathbf{n}} = \cos\theta\cos\phi\,\hat{\boldsymbol{d}}_s - \cos\theta\sin\phi\,\hat{\boldsymbol{d}}_{s_\perp} + \sin\theta\,\hat{\boldsymbol{H}}_s \tag{18}$$

where $\hat{\boldsymbol{d}}_{s_\perp} = \widetilde{\hat{\boldsymbol{H}}}_s \cdot \hat{\boldsymbol{d}}_s$.

Simulations have been carried out for a range of HAMR objects, released from the Laplace plane graveyard orbit with effective area-to-mass ratios from 0 up to 40 m^2/kg. The SRP acceleration, Eq. 17, was averaged over the object's unperturbed orbit and included in the averaged model developed in [20]. The angle θ in Eq. 18 was varied between 0° and 90° and ϕ assumed a value between $-45°$ and 45°. The details and results of these case studies will be presented elsewhere. For the object with $\phi = 0°$, the evolution in the (i, Ω) phase space is similar to what was predicted using the cannonball model [21]. When the SRP acceleration direction is tilted out of the ecliptic plane, the symmetry of the motion about the modified Laplace plane is eradicated for large values of ϕ. Nevertheless, in all cases considered, these HAMR objects if started from this inclined graveyard orbit never crossed through zero inclination over a hundred year evolution. Moreover, only the objects that are in near resonance with the Saros [20] (17 m^2/kg $\leq (1+\rho)A/m \leq$ 25 m^2/kg) came within 5° of the geostationary orbit.

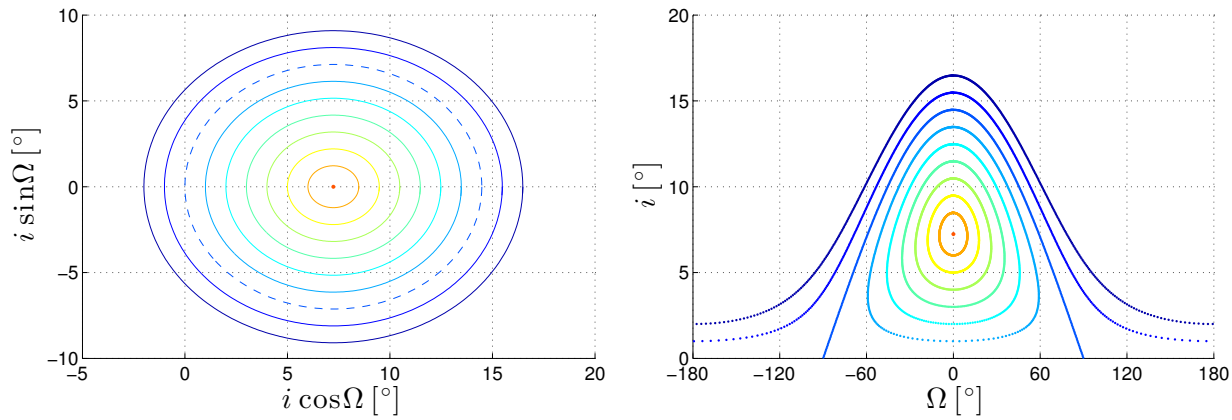
4. ECONOMIC VIABILITY AND ALTERNATIVE DISPOSAL OPTION

The current disposal orbits, Fig. 4, free desirable longitudinal positions for replacement satellites and reduce immediate collision hazards in GEO with an acceptable cost-to-benefit ratio, an important criterion. The cost in terms of incremental velocity is not more than 3.65 m/s per 100 km increase in altitude, which amounts to the fuel needed for one month operational station-keeping [7]. To place a satellite into the Laplace plane graveyard, not only must the current practice be implemented to remove the satellite from GEO altitude, but the satellite's orbit must be inclined by about 7.2°. This expensive plane change maneuver requires an incremental velocity of roughly 388 m/s, a third the cost of placing the satellite on an escape trajectory (for comparison, a de-orbit maneuver at GEO requires ∼1.5 km/s and an Earth-escape maneuver costs ∼1.2 km/s [18]). The IADC guidelines for storage orbits are based on a one-dimensional problem: define a safe minimum re-orbiting distance above GEO needed to isolate the retired satellites from GEO [8]. It was noted, however, that these guidelines should be updated as new information becomes available regarding the space environment, as to define what constitutes an effective graveyard [8].

An alternative disposal option to the Laplace plane graveyard is to simply incline the current disposal orbits by a minimum inclination, as shown in Fig. 7, such that the HAMR objects released here will not cross through the geostationary orbit. The basic physical principle behind this two-dimensional disposal scheme is easy to grasp. The orbit's angular momentum vector sweeps out a circular cone around the fixed pole of the classical Laplace plane, with the radius of the circular path being a function of the orbit's inclination to this fixed plane (Fig. 7(a)). Accordingly, tilting the orbit normal towards the axis of the Laplace plane shrinks this circle, and as the modified Laplace plane is further inclined, the HAMR objects released here will not cross below the initial inclination of the disposal orbit.

5. CONCLUSIONS

The importance of managing space debris is acknowledged by all space-faring nations, as the long-term financial, legal, and envionrmental implications of collisions between high-value satellites are manifest. The discovery of the



(a) Evolution as viewed from Earth's polar axis.

(b) Structure of the evolution in inclination and node phase space.

Fig. 7. Qualitative evolution, over 55 years, of the orbital planes of satellites released from inclined super-synchronous orbits, showing how the alternative graveyard orbit scheme would keep objects out of the equatorial plane. The dashed curve in Fig. 7(a) and the separatrix-like curve in Fig. 7(b) are the plane precession of a satellite released from zero inclination. The classical Laplace plane equilibrium is dot in the center of these plots.

high area-to-mass ratio debris reveals that the situation in the geostationary orbit region is even more critical than previously thought, and is becoming as compelling a problem as in LEO. As we begin to discover the full scope of the debris problem in GEO, we are finding that the current graveyard and mitigation practices are now obsolescent. We have used our understanding of the classical and modified Laplace planes for the identification of robust GEO disposal orbits. In accordance with Friesen et al. (1993), we propose the use of the stable Laplace plane equilibrium as a long-term graveyard for GEO. Based on our analysis of the modified Laplace plane, we showed that if satellites located in the classical Laplace plane graveyard orbit shed HAMR objects, their orbits will be trapped in (i, Ω) phase space, and will not likely cross through geostationary orbit. Thus, the Laplace plane graveyard orbit is analogous to the ocean gyres which are characterized by exceptionally high concentrations of marine debris that have been trapped by the currents [14]. We showed that our solutions are robust to more realistic SRP models, and proposed a cost-effective graveyard orbit based on the structure of the orbit plane evolution in the (i, Ω) phase space. Future trade space studies are needed to determine the minimum inclination that will yield the optimum cost-to-benefit ratio, and innovative methods that takes advantage of the natural forces to effect the plane change are desired.

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